VISIT…

EPILOGUE

How does the infinitesimal calculus as developed in this book relate to the traditional (or ε, δ) calculus? To get the proper perspective we shall sketch the history of the calculus.

Many problems involving slopes, areas, and volumes, which we would today call calculus problems, were solved by the ancient Greek mathematicians. The greatest of them was Archimedes (287–212 B.C.). Archimedes anticipated both the infinitesimal and the ε, δ approach to calculus. He sometimes discovered his results by reasoning with infinitesimals, but always published his proofs using the "method of exhaustion," which is similar to the ε, δ approach.

Calculus problems became important in the early 1600's with the development of physics and astronomy. The basic rules for differentiation and integration were discovered in that period by informal reasoning with infinitesimals. Kepler, Galileo, Fermat, and Barrow were among the contributors.

In the 1660's and 1670's Sir Isaac Newton and Gottfried Wilhelm Leibniz independently "invented" the calculus. They took the major step of recognizing the importance of a collection of isolated results and organizing them into a whole.

Newton, at different times, described the derivative of y (which he called the "fluxion" of y) in three different ways, roughly

(1) The ratio of an infinitesimal change in y to an infinitesimal change in x. (The infinitesimal method.)

(2) The limit of the ratio of the change in y to the change in x, $\Delta y/\Delta x$, as Δx approaches zero. (The limit method.)

(3) The velocity of y where x denotes time. (The velocity method.)

In his later writings Newton sought to avoid infinitesimals and emphasized the methods (2) and (3).

Leibniz rather consistently favored the infinitesimal method but believed (correctly) that the same results could be obtained using only real numbers. He regarded the infinitesimals as "ideal" numbers like the imaginary numbers. To justify them he proposed his law of continuity: "In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the terminus may

also be included."[1] This "law" is far too imprecise by present standards. But it was a remarkable forerunner of the Transfer Principle on which modern infinitesimal calculus is based. Leibniz was on the right track, but 300 years too soon!

The notation developed by Leibniz is still in general use today, even though it was meant to suggest the infinitesimal method: dy/dx for the derivative (to suggest an infinitesimal change in y divided by an infinitesimal change in x), and $\int_b^a f(x)\,dx$ for the integral (to suggest the sum of infinitely many infinitesimal quantities $f(x)\,dx$).

All three approaches had serious inconsistencies which were criticized most effectively by Bishop Berkeley in 1734. However, a precise treatment of the calculus was beyond the state of the art at the time, and the three intuitive descriptions (1)–(3) of the derivative competed with each other for the next two hundred years. Until sometime after 1820, the infinitesimal method (1) of Leibniz was dominant on the European continent, because of its intuitive appeal and the convenience of the Leibniz notation. In England the velocity method (3) predominated; it also has intuitive appeal but cannot be made rigorous.

In 1821 A. L. Cauchy published a forerunner of the modern treatment of the calculus based on the limit method (2). He defined the integral as well as the derivative in terms of limits, namely

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0^+} \sum_a^b f(x)\,\Delta x.$$

He still used infinitesimals, regarding them as variables which approach zero. From that time on, the limit method gradually became the dominant approach to calculus, while infinitesimals and appeals to velocity survived only as a manner of speaking. There were two important points which still had to be cleared up in Cauchy's work, however. First, Cauchy's definition of limit was not sufficiently clear; it still relied on the intuitive use of infinitesimals. Second, a precise definition of the real number system was not yet available. Such a definition required a better understanding of the concepts of set and function which were then evolving.

A completely rigorous treatment of the calculus was finally formulated by Karl Weierstrass in the 1870's. He introduced the ε, δ condition as the definition of limit. At about the same time a number of mathematicians, including Weierstrass, succeeded in constructing the real number system from the positive integers. The problem of constructing the real number system also led to development of set theory by Georg Cantor in the 1870's. Weierstrass' approach has become the traditional or "standard" treatment of calculus as it is usually presented today. It begins with the ε, δ condition as the definition of limit and goes on to develop the calculus entirely in terms of the real number system (with no mention of infinitesimals). However, even when calculus is presented in the standard way, it is customary to argue informally in terms of infinitesimals, and to use the Leibniz notation which suggests infinitesimals.

From the time of Weierstrass until very recently, it appeared that the limit method (2) had finally won out and the history of elementary calculus was closed. But in 1934, Thoralf Skolem constructed what we here call the hyperintegers and proved that the analogue of the Transfer Principle holds for them. Skolem's construction (now called the ultraproduct construction) was later extended to a wide class of structures, including the construction of the hyperreal numbers from the real numbers.

[1] See Kline, p. 385.

The name "hyperreal" was first used by E. Hewitt in a paper in 1948. The hyperreal numbers were known for over a decade before they were applied to the calculus.

Finally in 1961 Abraham Robinson discovered that the hyperreal numbers could be used to give a rigorous treatment of the calculus with infinitesimals. The presentation of the calculus which was given in this book is based on Robinson's treatment (but modified to make it suitable for a first course).

Robinson's calculus is in the spirit of Leibniz' old method of infinitesimals. There are major differences in detail. For instance, Leibniz defined the derivative as the ratio $\Delta y/\Delta x$ where Δx is infinitesimal, while Robinson defines the derivative as the *standard part* of the ratio $\Delta y/\Delta x$ where Δx is infinitesimal. This is how Robinson avoids the inconsistencies in the old infinitesimal approach. Also, Leibniz' vague law of continuity is replaced by the precisely formulated Transfer Principle.

The reason Robinson's work was not done sooner is that the Transfer Principle for the hyperreal numbers is a type of axiom that was not familiar in mathematics until recently. It arose in the subject of model theory, which studies the relationship between axioms and mathematical structures. The pioneering developments in model theory were not made until the 1930's, by Gödel, Malcev, Skolem, and Tarski; and the subject hardly existed until the 1950's.

Looking back we see that the method of infinitesimals was generally preferred over the method of limits for over 150 years after Newton and Leibniz invented the calculus, because infinitesimals have greater intuitive appeal. But the method of limits was finally adopted around 1870 because it was the first mathematically precise treatment of the calculus. Now it is also possible to use infinitesimals in a mathematically precise way. Infinitesimals in Robinson's sense have been applied not only to the calculus but to the much broader subject of analysis. They have led to new results and problems in mathematical research. Since Skolem's infinite hyperintegers are usually called nonstandard integers, Robinson called the new subject "nonstandard analysis." (He called the real numbers "standard" and the other hyperreal numbers "nonstandard." This is the origin of the name "standard part.")

The starting point for this course was a pair of intuitive pictures of the real and hyperreal number systems. These intuitive pictures are really only rough sketches that are not completely trustworthy. In order to be sure that the results are correct, the calculus must be based on mathematically precise descriptions of these number systems, which fill in the gaps in the intuitive pictures. There are two ways to do this. The quickest way is to list the mathematical properties of the real and hyperreal numbers. These properties are to be accepted as basic and are called *axioms*. The second way of mathematically describing the real and hyperreal numbers is to start with the positive integers and, step by step, construct the integers, the rational numbers, the real numbers, and the hyperreal numbers. This second method is better because it shows that there really is a structure with the desired properties. At the end of this epilogue we shall briefly outline the construction of the real and hyperreal numbers and give some examples of infinitesimals.

We now turn to the first way of mathematically describing the real and hyperreal numbers. We shall list two groups of axioms in this epilogue, one for the real numbers and one for the hyperreal numbers. The axioms for the hyperreal numbers will just be more careful statements of the Extension Principle and Transfer Principle of Chapter 1. The axioms for the real numbers come in three sets: the Algebraic Axioms, the Order Axioms, and the Completeness Axiom. All the familiar facts about the real numbers can be proved using only these axioms.

I. ALGEBRAIC AXIOMS FOR THE REAL NUMBERS

A Closure laws 0 *and* 1 *are real numbers. If* a *and* b *are real numbers, then so are* $a + b$, ab, *and* $-a$. *If* a *is a real number and* $a \neq 0$, *then* $1/a$ *is a real number.*

B Commutative laws $a + b = b + a \qquad ab = ba.$

C Associative laws $a + (b + c) = (a + b) + c \qquad a(bc) = (ab)c.$

D Identity laws $0 + a = a \qquad 1 \cdot a = a.$

E Inverse laws $a + (-a) = 0 \qquad \text{If } a \neq 0, \quad a \cdot \frac{1}{a} = 1.$

F Distributive law $a \cdot (b + c) = ab + ac.$

DEFINITION

The ***positive integers*** *are the real numbers* $1, 2 = 1 + 1, 3 = 1 + 1 + 1, 4 = 1 + 1 + 1 + 1$, *and so on.*

II. ORDER AXIOMS FOR THE REAL NUMBERS

A $0 < 1.$

B Transitive law *If* $a < b$ *and* $b < c$ *then* $a < c.$

C Trichotomy law *Exactly one of the relations* $a < b, a = b, b < a$, *holds.*

D Sum law *If* $a < b$, *then* $a + c < b + c.$

E Product law *If* $a < b$ *and* $0 < c$, *then* $ac < bc.$

F Root axiom *For every real number* $a > 0$ *and every positive integer* n, *there is a real number* $b > 0$ *such that* $b^n = a.$

III. COMPLETENESS AXIOM

Let A *be a set of real numbers such that whenever* x *and* y *are in* A, *any real number between* x *and* y *is in* A. *Then* A *is an interval.*

THEOREM

An increasing sequence $\langle S_n \rangle$ *either converges or diverges to* ∞.

PROOF Let T be the set of all real numbers x such that $x \leq S_n$ for some n. T is obviously nonempty.

Case 1 T is the whole real line. If H is infinite we have $x \leq S_H$ for all real numbers x. So S_H is positive infinite and $\langle S_n \rangle$ diverges to ∞.

Case 2 T is not the whole real line. By the Completeness Axiom, T is an interval $(-\infty, b]$ or $(-\infty, b)$. For each real $x < b$, we have

$$x \leq S_n \leq S_{n+1} \leq S_{n+2} \leq \cdots \leq b$$

for some n. It follows that for infinite H, $S_H \leq b$ and $S_H \approx b$. Therefore $\langle S_n \rangle$ converges to b.

We now take up the second group of axioms, which give the properties of the hyperreal numbers. There will be two axioms, called the Extension Axiom and the Transfer Axiom, which correspond to the Extension Principle and Transfer Principle of Section 1.5. We first state the Extension Axiom.

I*. EXTENSION AXIOM

(a) *The set R of real numbers is a subset of the set R^* of hyperreal numbers.*

(b) *There is given a relation $<^*$ on R^*, such that the order relation $<$ on R is a subset of $<^*$, $<^*$ is transitive ($a <^* b$ and $b <^* c$ implies $a <^* c$), and $<^*$ satisfies the Trichotomy Law: for all a, b in R^*, exactly one of $a <^* b$, $a = b$, $b <^* a$ holds.*

(c) *There is a hyperreal number ε such that $0 <^* \varepsilon$ and $\varepsilon <^* r$ for each positive real number r.*

(d) *For each real function f, there is given a hyperreal function f^* with the same number of variables, called the natural extension of f.*

Part (c) of the Extension Axiom states that there is at least one positive infinitesimal. Part (d) gives us the natural extension for each real function. The Transfer Axiom will say that this natural extension has the same properties as the original function.

Recall that the Transfer Principle of Section 1.5 made use of the intuitive idea of a real statement. Before we can state the Transfer Axiom, we must give an exact mathematical explanation of the notion of a real statement. This will be done in several steps, first introducing the concepts of a real expression and a formula.

We begin with the concept of a real *expression*, or *term*, built up from variables and real constants using real functions. Real expressions can be built up as follows:

(1) A real constant standing alone is a real expression.

(2) A variable standing alone is a real expression.

(3) If e is a real expression, and f is a real function of one variable, then $f(e)$ is a real expression. Similarly, if $e_1, \ldots, e_n$ are real expressions, and g is a real function of n variables, then $g(e_1, \ldots, e_n)$ is a real expression.

Step (3) can be used repeatedly to build up longer expressions. Here are some examples of real expressions, where x and y are variables:

$$2, \quad x + y, \quad |x - 4|, \quad \sin(\pi y^2), \quad \frac{\sqrt{x} + \sqrt{y}}{\sqrt{3}}, \quad g(x, f(0)), \quad 1/0.$$

By a *formula*, we mean a statement of one of the following kinds, where d and e are real expressions:

(1) An *equation* between two real expressions, $d = e$.

(2) An *inequality* between two real expressions, $d < e$, $d \leq e$, $d > e$, $d \geq e$, or $d \neq e$.

(3) A statement of the form "e is defined" or "e is undefined."

Here are some examples of formulas:

$$x + y = 5,$$
$$f(x) = \frac{1 - x^2}{1 + x},$$
$$g(x, y) < f(t),$$
$f(x, x)$ is undefined.

If each variable in a formula is replaced by a real number, the formula will be either true or false. Ordinarily, a formula will be true for some values of the variables and false for others. For example, the formula $x + y = 5$ will be true when $(x, y) = (4, 1)$ and false when $(x, y) = (7, -2)$.

DEFINITION

*A **real statement** is either a nonempty finite set of formulas T or a combination involving two nonempty finite sets of formulas S and T that states that "whenever every formula in S is true, every formula in T is true."*

We shall give several comments and examples to help make this definition clear. Sometimes, instead of writing "whenever every formula in S is true, every formula in T is true" we use the shorter form "if S then T" for a real statement. Each of the Algebraic Axioms for the Real Numbers is a real statement. The commutative laws, associative laws, identity laws, and distributive laws are real statements. For example, the commutative laws are the pair of formulas

$$a + b = b + a, \qquad ab = ba,$$

which involve the two variables a and b. The closure laws may be expressed as four real statements:

$a + b$ is defined,
ab is defined,
$-a$ is defined,
if $a \neq 0$, then $1/a$ is defined.

The inverse laws consist of two more real statements. The Trichotomy Law is part of the Extension Axiom, and all of the other Order Axioms for the Real Numbers are real statements. However, the Completeness Axiom is not a real statement, because it is not built up from equations and inequalities between terms.

A typical example of a real statement is the inequality for exponents discussed in Section 8.1:

$$\text{if } a \geq 0, \text{ and } q \geq 1, \text{ then } (a + 1)^q \geq aq + 1.$$

This statement is true for all real numbers a and q.

A formula can be given a meaning in the hyperreal number system as well as in the real number system. Consider a formula with the two variables x and y. When x and y are replaced by particular real numbers, the formula will be either true or false in the real number system. To give the formula a meaning in the hyperreal number system, we replace each real function by its natural extension and replace

the real order relation $<$ by the hyperreal relation $<^*$. When x and y are replaced by hyperreal numbers, each real function f is replaced by its natural extension f^*, and the real order relation $<$ is replaced by $<^*$, the formula will be either true or false in the hyperreal number system.

For example, the formula $x + y = 5$ is true in the hyperreal number system when $(x, y) = (2 - \varepsilon, 3 + \varepsilon)$, but false when $(x, y) = (2 + \varepsilon, 3 + \varepsilon)$, if ε is nonzero.

We are now ready to state the Transfer Axiom.

II*. TRANSFER AXIOM

Every real statement that holds for all real numbers holds for all hyperreal numbers.

It is possible to develop the whole calculus course as presented in this book from these axioms for the real and hyperreal numbers. By the Transfer Axiom, all the Algebraic Axioms for the Real Numbers also hold true for the hyperreal numbers. In other words, we can transfer every Algebraic Axiom for the real numbers to the hyperreal numbers. We can also transfer every Order Axiom for the real numbers to the hyperreal numbers. The Trichotomy Law is part of the Extension Axiom. Each of the other Order Axioms is a real statement and thus carries over to the hyperreal numbers by the Transfer Axiom. Thus we can make computations with the hyperreal numbers in the same way as we do for the real numbers.

There is one fact of basic importance that we state now as a theorem.

THEOREM (Standard Part Principle)

For every finite hyperreal number b, there is exactly one real number r that is infinitely close to b.

PROOF We first show that there cannot be more than one real number infinitely close to b. Suppose r and s are real numbers such that $r \approx b$ and $s \approx b$. Then $r \approx s$, and since r and s are real, r must be equal to s. Thus there is at most one real number infinitely close to b.

We now show that there is a real number infinitely close to b. Let A be the set of all real numbers less than b. Then any real number between two elements of A is an element of A. By the Completeness Axiom for the real numbers, A is an interval. Since the hyperreal number b is finite. A must be an interval of the form $(-\infty, r)$ or $(-\infty, r]$ for some real number r. Every real number $s < r$ belongs to A, so $s < b$. Also, every real number $t > r$ does not belong to A, so $t \geq b$. This shows that r is infinitely close to b.

It was pointed out earlier that the Completeness Axiom does not qualify as a real statement. For this reason, the Transfer Principle cannot be used to transfer the Completeness Axiom to the hyperreal numbers. In fact, the Completeness Axiom is *not* true for the hyperreal numbers. By a *closed hyperreal interval*, we mean a set of hyperreal numbers of the form $[a, b]$, the set of all hyperreal numbers x for which $a \leq x \leq b$, where a and b are hyperreal constants. Open and half-open hyperreal intervals are defined in a similar way. When we say that the Completeness Axiom is not true for the hyperreal numbers, we mean that there actually are sets A of hyperreal numbers such that:

(a) Whenever x and y are in A, any hyperreal number between x and y is in A.

(b) A is not a hyperreal interval.

Here are two quite familiar examples.

EXAMPLE 1 The set A of all infinitesimals has property (a) above but is not a hyperreal interval. It has property (a) because any hyperreal number that is between two infinitesimals is itself infinitesimal. We show that A is not a hyperreal interval. A cannot be of the form $[a, \infty)$ or (a, ∞) because every infinitesimal is less than 1. A cannot be of the form $[a, b]$ or $(a, b]$, because if b is positive infinitesimal, then $2 \cdot b$ is a larger infinitesimal. A cannot be of the form $[a, b)$ or (a, b), because if b is positive and not infinitesimal, then $b/2$ is less than b but still positive and not infinitesimal.

The set B of all finite hyperreal numbers is another example of a set that has property (a) above but is not an interval.

Here are some examples that may help to illustrate the nature of the hyperreal number system and the use of the Transfer Axiom.

EXAMPLE 2 Let f be the real function given by the equation

$$f(x) = \sqrt{1 - x^2}.$$

Its graph is the unit semicircle with center at the origin. The following two real statements hold for all real numbers x:

$$\text{whenever } 1 - x^2 \geq 0, \qquad f(x) = \sqrt{1 - x^2};$$
$$\text{whenever } 1 - x^2 < 0, \qquad f(x) \text{ is undefined.}$$

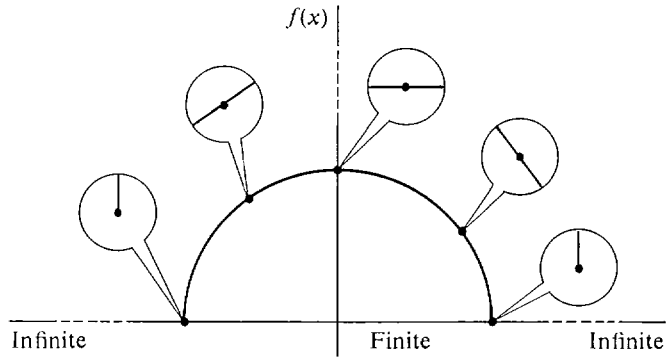
By the Transfer Axiom, these real statements also hold for all hyperreal numbers x. Therefore the natural extension f^* of f is given by the same equation

$$f^*(x) = \sqrt{1 - x^2}.$$

The domain of f^* is the set of all hyperreal numbers between -1 and 1. The hyperreal graph of f^*, shown in Figure E.1, can be drawn on paper by drawing the real graph of $f(x)$ and training an infinitesimal microscope on certain key points.

EXAMPLE 3 Let f be the identity function on the real numbers, $f(x) = x$. By the Transfer Axiom, the equation $f(x) = x$ is true for all hyperreal x. Thus the natural extension f^* of f is defined, and $f^*(x) = x$ for all hyperreal x. Figure E.2 shows the hyperreal graph of f^*. Under a microscope, it has a 45° slope.

Here is an example of a hyperreal function that is not the natural extension of a real function.



Hyperreal graph of $f(x) = \sqrt{1 - x^2}$

Figure E.1

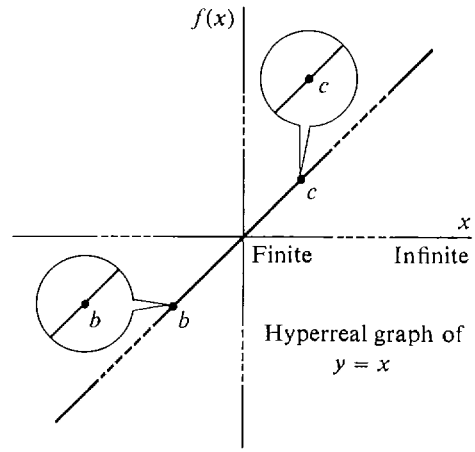


Figure E.2

EXAMPLE 4 One hyperreal function, which we have already studied in some detail, is the *standard part function* $st(x)$. For real numbers the standard part function has the same values as the identity function,

$$st(x) = x \quad \text{for all real } x.$$

However, the hyperreal graph of $st(x)$, shown in Figure E.3, is very different from the hyperreal graph of the identity function f^*. The domain of the standard part function is the set of all finite numbers, while f^* has domain R^*. Thus for infinite x, $f^*(x) = x$, but $st(x)$ is undefined. If x is finite but not real, $f^*(x) = x$ but $st(x) \neq x$. Under the microscope, an infinitesimal piece of the graph of the standard part function is horizontal, while the identity function has a 45° slope.

The standard part function is not the natural extension of the identity function, and hence is not the natural extension of any real function.

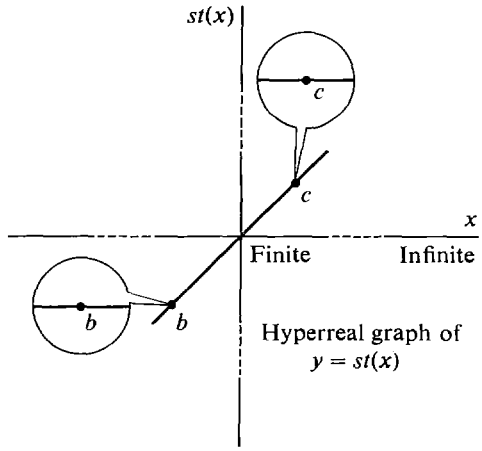


Figure E.3

The standard part function is the only hyperreal function used in this course except for natural extensions of real functions.

We conclude with a few words about the construction of the real and the hyperreal numbers. Before Weierstrass, the rational numbers were on solid ground but the real numbers were something new. Before one could safely use the axioms for the real numbers, it had to be shown that the axioms did not lead to a contradiction. This was done by starting with the rational numbers and constructing a structure which satisfied all the axioms for the real numbers. Since anything proved from the axioms is true in this structure, the axioms cannot lead to a contradiction.

The idea is to construct *real* numbers out of Cauchy sequences of *rational* numbers.

DEFINITION

A Cauchy Sequence is a sequence $\langle a_1, a_2, \ldots \rangle$ *of numbers such that for every real* $\varepsilon > 0$ *there is an integer* n_ε *such that the numbers*

$$\langle a_{n_\varepsilon}, a_{n_\varepsilon+1}, a_{n_\varepsilon+2}, \ldots \rangle$$

are all within ε *of each other.*

Two Cauchy sequences

$$\langle a_1, a_2, \ldots \rangle, \qquad \langle b_1, b_2, \ldots \rangle$$

of rational numbers are called *Cauchy equivalent*, in symbols $\langle a_1, a_2, \ldots \rangle \equiv \langle b_1, b_2, \ldots \rangle$, if the difference sequence

$$\langle a_1 - b_1, a_2 - b_2, \ldots \rangle$$

converges to zero. (Intuitively this means that the two sequences have the same limit.)

PROPERTIES OF CAUCHY EQUIVALENCE

(1) If $\langle a_1, a_2, \ldots \rangle \equiv \langle a'_1, a'_2, \ldots \rangle$ and $\langle b_1, b_2, \ldots \rangle \equiv \langle b'_1, b'_2, \ldots \rangle$

then the sum sequences are equivalent,

$$\langle a_1 + b_1, a_2 + b_2, \ldots \rangle \equiv \langle a'_1 + b'_1, a'_2 + b'_2, \ldots \rangle.$$

(2) Under the same hypotheses, the product sequences are equivalent,

$$\langle a_1 \cdot b_1, a_2 \cdot b_2, \ldots \rangle \equiv \langle a'_1 \cdot b'_1, a'_2 \cdot b'_2, \ldots \rangle.$$

(3) If $a_n = b_n$ for all but finitely many n, then

$$\langle a_1, a_2, \ldots \rangle \equiv \langle b_1, b_2, \ldots \rangle.$$

The set of real numbers is then defined as the set of all equivalence classes of Cauchy sequences of rational numbers. A rational number r corresponds to the equivalence class of the constant sequence $\langle r, r, r, \ldots \rangle$. The sum of the equivalence class of $\langle a_1, a_2, \ldots \rangle$ and the equivalence class of $\langle b_1, b_2, \ldots \rangle$ is defined as the equivalence class of the sum sequence

$$\langle a_1 + b_1, a_2 + b_2, \ldots \rangle.$$

The product is defined in a similar way. It can be shown that all the axioms for the real numbers hold for this structure.

Today the real numbers are on solid ground and the hyperreal numbers are a new idea. Robinson used the ultraproduct construction of Skolem to show that the axioms for the hyperreal numbers (for example, as used in this book) do not lead to a contradiction. The method is much like the construction of the real numbers from the rationals. But this time the real number system is the starting point. We construct *hyperreal* numbers out of arbitrary (not just Cauchy) sequences of *real* numbers.

By an *ultraproduct equivalence* we mean an equivalence relation $\equiv$ on the set of all sequences of real numbers which have the properties of Cauchy equivalence (1)–(3) and also

(4) If each a_n belongs to the set $\{0, 1\}$ then $\langle a_1, a_2, \ldots \rangle$ is equivalent to exactly one of the constant sequences $\langle 0, 0, 0, \ldots \rangle$ or $\langle 1, 1, 1, \ldots \rangle$.

Given an ultraproduct equivalence relation, the set of hyperreal numbers is defined as the set of all equivalence classes of sequences of real numbers. A real number r corresponds to the equivalence class of the constant sequence $\langle r, r, r, \ldots \rangle$. Sums and products are defined as for Cauchy sequences. The natural extension f^* of a real function $f(x)$ is defined so that the image of the equivalence class of $\langle a_1, a_2, \ldots \rangle$ is the equivalence class of $\langle f(a_1), f(a_2), \ldots \rangle$. It can be proved that ultraproduct equivalence relations exist, and that all the axioms for the real and hyperreal numbers hold for the structure defined in this way.

When hyperreal numbers are constructed as equivalence classes of sequences of real numbers, we can give specific examples of infinite hyperreal numbers. The equivalence class of

$$\langle 1, 2, 3, \ldots, n, \ldots \rangle$$

is a positive infinite hyperreal number. The equivalence class of

$$\langle 1, 4, 9, \ldots, n^2, \ldots \rangle$$

is larger, and the equivalence class of

$$\langle 1, 2, 4, \ldots, 2^n, \ldots \rangle$$

is a still larger infinite hyperreal number.

We can also give examples of nonzero infinitesimals. The equivalence classes of

$$\langle 1, 1/2, 1/3, \ldots, 1/n, \ldots \rangle,$$

$$\langle 1, 1/4, 1/9, \ldots, n^{-2}, \ldots \rangle,$$

and

$$\langle 1, 1/2, 1/4, \ldots, 2^{-n}, \ldots \rangle$$

are progressively smaller positive infinitesimals.

The mistake of Leibniz and his contemporaries was to identify all the infinitesimals with zero. This leads to an immediate contradiction because dy/dx becomes 0/0. In the present treatment the equivalence classes of

$$\langle 1, 1/2, 1/3, \ldots, 1/n, \ldots \rangle$$

and

$$\langle 0, 0, 0, \ldots, 0, \ldots \rangle$$

are different hyperreal numbers. They are not equal but merely have the same standard part, zero. This avoids the contradiction and once again makes infinitesimals a mathematically sound method.

For more information about the ideas touched on in this epilogue, see the instructor's supplement, *Foundations of Infinitesimal Calculus*, which has a self-contained treatment of ultraproducts and the hyperreal numbers.

FOR FURTHER READING ON THE HISTORY OF THE CALCULUS SEE:

The History of the Calculus and its Conceptual Development; Carl C. Boyer, Dover, New York, 1959.

Mathematical Thought from Ancient to Modern Times; Morris Kline, Oxford Univ. Press, New York, 1972.

Non-standard Analysis; Abraham Robinson, North-Holland, Amsterdam, London, 1966.

FOR ADVANCED READING ON INFINITESIMAL ANALYSIS SEE NON-STANDARD ANALYSIS BY ABRAHAM ROBINSON AND:

Lectures on Non-standard Analysis; M. Machover and J. Hirschfeld, Springer-Verlag, Berlin, Heidelberg, New York, 1969.

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